MTH 507 (or 605): Introduction to algebraic topology (or Topology I)

Semester 1, 2015-16

1. Point-set topology

1.1. **Topological space.**

(i) Topology on a set with example.
(ii) Coarser and finer topologies.

1.2. **Basis for a topological space.**

(i) Basis for a topology with examples.
(ii) Let $\mathcal{B}$ be the basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.
(iii) Suppose that $\mathcal{B}$ is a collection of open sets of a topological X such that for each open set U of X and each $x \in U$, there is an element $B \in \mathcal{B}$ such that $x \in B \subset U$. Then $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X.
(iv) Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:
   (a) $\mathcal{T}$ is finer than $\mathcal{T}'$.
   (b) For each $x \in X$, and each basis element $B \in \mathcal{B}$ containing x, there exists a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.
(v) The standard, lower-limit ($\mathbb{R}_\ell$), and K ($\mathbb{R}_K$) topologies on $\mathbb{R}$.

1.3. **Subspace topology.**

(i) Subspaces with examples.
(ii) If $\mathcal{B}$ is a basis for a topology on X, then the collection $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.
(iii) If U is open in a set Y and Y is open in X, then U is open in X.

1.4. **Closed sets and limit points.**

(i) Closed sets with examples.
(ii) Let X be a topological space. Then
   (a) $\emptyset$ and X are closed.
   (b) Arbitrary intersection of closed sets is closed.
   (c) Finite union of closed sets is closed.
(iii) Let Y be a subspace of X. Then a set A is closed in Y if, and only if it is the intersection of a closed set in X with Y.
(iv) Closure ($\bar{A}$) and interior (A°) of set A .

(v) Let A be a subset of Y. Then the closure of A in Y is the closure of A in X intersected with Y.

(vi) Let A be a subset of X. Then $X \in \bar{A}$ if and only if every open set containing x intersects A.

(vii) Limit points and set A' of all limit points of A.

(viii) Let A be a subset of X. Then $\bar{A} = A \cup A'$.

(ix) Hausdorff and T_1 spaces with examples.

(x) Let A be a subset of a T_1 space X. Then x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

(xi) Any sequence converges to a unique limit point in a Hausdorff space.

(xii) The product of two Hausdorff spaces is Hausdorff, and a subspace of a Hausdorff space is Hausdorff.

1.5. **Continuous functions with examples.**

(i) Continuous functions with examples.

(ii) Let X and Y be topological spaces, and let $f : X \to Y$. Then the following are equivalent:

(a) f is continuous.

(b) For every subset $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$.

(c) For every closed subset B of Y, $f^{-1}(B)$ is closed in X.

(iii) Homeomorphisms with examples.

(iv) A map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|_{U_\alpha}$ is continuous for each α.

(v) Pasting Lemma: Let $X = A \cup B$, where A and B are closed in X, and let $f : A \to X$, $g : B \to X$ be continuous. If $f(x) = g(x)$ for each $x \in A \cap B$, then the function h defined by

$$h(x) = \begin{cases} f(x), & \text{if } x \in A \\ g(x), & \text{if } x \in B \end{cases}$$

is continuous.

1.6. **Product topology.**

(i) Product and box topologies on $\prod_\alpha X_\alpha$, and their bases.

(ii) If each X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.

(iii) Let $\{X_\alpha\}$ be a family of topological spaces, and let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given either the product or the box topology, then

$$\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$$

.

(iv) If $f : A \to \prod X_\alpha$ be given by the equation $f(a) = (f_\alpha(a))$, where $f_\alpha : A \to X_\alpha$ for each α, and $\prod X_\alpha$ has the product topology. Then the function f is continuous if and only if each function f_α is continuous.

1.7. **Metric topology.**

(i) Metric topology with examples, and metrizability.
(ii) Sequence Lemma: Let X be a topological, and let $A \subset X$. If there exists a sequence of points in A coverging to x, then $x \in \bar{A}$. The converse holds if X is metrizable.
(iii) If $f : X \to Y$ is continuous, then for every convergent sequence $x_n \to x$, the sequence $(f(x_n))$ converges to $f(x)$. The converse holds if X is metrizable.
(iv) Uniform convergence of a sequence of functions.
(v) Uniform limit theorem: Let $f_n : X \to Y$ be a sequence of continuous functions from a topological space X to a metric space Y. If (f_n) converges uniformly to f, then f is continuous.
(vi) The box topology on $\mathbb{R}^\infty$ and any uncountable product of $\mathbb{R}$ with itself are not metrizable.

1.8. **Quotient topology.**

(i) Quotient map with examples.
(ii) Saturated subsets.
(iii) Open and closed maps.
(iv) Quotient topology and quotient space.
(v) Let $p : X \to Y$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a map that is constant on each fiber $p^{-1}(\{y\})$. Then g induces a map $f : Y \to Z$ such that $f \circ p = g$. The induced map f is continuous if and only if g is continuous. Moreover, f is a quotient map if and only if g is a quotient map.
(vi) Let $g : X \to Z$ be a surjective continuous map. Let $X^* = \{g^{-1}(\{z\}) \mid z \in Z\}$ with the quotient topology.
  (a) The map g induces a bijective map $f : X^* \to Z$, which is a homeomorphism if and only if g is a quotient map.
  (b) If Z is Hausdorff, so is X^*.

1.9. **Connected and path connected spaces.**

(i) Connected spaces with examples.

(ii) Union of a collection of connected subspaces of X that have a point in common is connected.
(iii) Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is also connected.
(iv) The image of a connected space under a continuous map is connected.
(v) A finite cartesian product of connected spaces is connected.
(vi) $\mathbb{R}^\infty$ is not connected with box topology, but connected with product topology.
(vii) The intervals in $\mathbb{R}$ are connected.
(viii) Path connectedness with examples.

1.10. **Components and local connectedness.**

(i) Connected and path components.
(ii) Local connectedness and local path connectedness with examples.
(iii) A space is locally connected if and only if for every open set U of X, each component of U is open in X.
(iv) A space X is locally path connected if and only if for every open set U of X, each path component of U is open in X.
(v) Each path component of a space X lies in a component of X. If X is locally path connected, then the components and path components of X are the same.

1.11. **Compact spaces.**

(i) Open covers.
(ii) Compact spaces with examples.
(iii) Compactness of a subpsace.
(iv) Every closed subspace of a compact space is compact.
(v) Every compact subspace of a Hausdorff space is closed.
(vi) Countinuous image of a compact space is compact.
(vii) Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.
(viii) Product of finitely many compact spaces is compact.
(ix) Finite intersection property.
(x) A space X is compact if and only if for every finite collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersections $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

1.12. **Compact subspaces of the real line.**

(i) Every closed interval in $\mathbb{R}$ is compact.
(ii) A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.
(iii) Lebesque number lemma: Let $\mathcal{A}$ be an open covering of the metric space (X, d). If X is compact, then there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing it.
(iv) Uniform continuity theorem: A continuous function on a compact metric space is uniformly continuous.
(v) A nonempty Hausdorff space with no isolated points is uncountable.
(vi) Every closed interval in $\mathbb{R}$ is uncountable.

1.13. **Limit point compactness.**

(i) Limit point compactness and sequential compactness with examples.
(ii) Compactness implies limit point compactness, but not conversely.
(iii) In a metrizable space X, the following are equivalent:
  (a) X is compact.
  (b) X is limit point compact.
  (c) X is sequentially compact.

1.14. **Local compactness.**

(i) Local compactness with examples.
(ii) X is a locally compact Hausdorff space if and only if there exists a space Y satisfying the following conditions.
  (a) X is a subspace of Y.
  (b) the set $Y - X$ is a single point.
  (c) Y is a compact Hausdorff space.

  If Y and Y' are two spaces satisfying these conditions, then there is a homepmorphism of Y with Y' that equals the identity on X.
(iii) One-point compactification with examples.
(iv) Let X be a Hausdorff space. Then X is locally compact if and only if given $x \in X$, and given a neighborhood $U \ni x$, there is a neighborhood $V \ni x$ such that $\bar{V}$ is compact and $\bar{V} \subset U$.
(v) Let X be a locally compact Hausdorff space. If A is either open or closed in X, then A is locally compact.

1.15. **Countability axioms.**

(i) First- and second-countability axioms.

(ii) If there is a sequence of points of $A \subset X$ converging to x, then $x \in \overline{A}$. The converse holds if x is first-countable.

(iii) If $f : X \to Y$ is continuous, then for every convergent sequence $x_n \to x$, the sequence $f(x_n) \to f(x)$. The converse holds if X is first-countable.

(iv) The subspace of a first-countable (or second-countable) space is first-countable (or second countable).

(v) The product of first-countable (or second-countable) spaces is first-countable (or second countable).

(vi) If f is second-countable, then

   (a) every open covering of X contains a countable subcovering.

   (b) there exists a countable subset of X that is dense in X.

(vii) Lindelöf space.

(viii) $\mathbb{R}_\ell$ is not second-countable.

(ix) Product of Lindelöf spaces need not be Lindelöf.

(x) Regular and normal spaces.

1.16. **Separation axioms.**

(i) The subspace of a Huasdorff space is Hausdorff. The product of Hausdorff spaces in Hausdorff.

(ii) The subspace of a regular space is regular. The product of regular spaces in regular.

(iii) $\mathbb{R}_k$ is Hausdorff but not regular.

(iv) $\mathbb{R}_\ell^2$ is not normal.

(v) Every regular space with countable basis is normal.

(vi) Every compact Hausdorff space is normal.

(vii) Every metrizable space is normal.

(viii) If J is uncountable, then the product space $\mathbb{R}^J$ is not normal.

(ix) (Urysohn's Lemma) Let X be a normal space, and let A and B be disjoint subsets of X. Then there exists a continuous map $f : X \to [a, b]$ such that $f(x) = a$ for every $x \in A$ and $f(x) = b$ for every $x \in B$.

(x) (Urysohn metrization theorem) Every regular second-countable space is metrizable.

(xi) (Tietze extension theorem) Let X be a normal space, and let A be a closed subspace of X.

   (a) Then any continuous map of A into $[a, b]$ may be extended to a continuous map from X into [a,b].

   (b) Any continuous map of A into $\mathbb{R}$ may be extended to a continuous map of X into $\mathbb{R}$.

(xii) m-manifold and partitions of unity.
(xiii) If $\{U_1, \ldots, U_n\}$ be a open covering of a normal space X, then there exists a partition of unity dominated by $\{U_i\}$.
(xiv) A compact m-manifold can be imbedded in $\mathbb{R}^n$ for some positive integer n.

1.17. **Nagata-Smirnov metrization.**

(i) Local finiteness and countable local finiteness.
(ii) Open and closed refinements.
(iii) Every open cover of a metrizable space has countably locally finite refinement.
(iv) (Nagata-Smirnov metrization theorem) A space X is metrizable if and only if X is regular and has a countably locally finite basis.

1.18. **Paracompactness.**

(i) Paracompactness.
(ii) Every paracompact Hausdorff space is normal.
(iii) Every closed subspace of a paracompact space is paracompact.
(iv) Every metrizable space is paracompact.
(v) Every Lindelöf space is paracompact.

2. Algebraic topology

2.1. **Homotopy.**

(i) Homotopy of continuous maps.
(ii) Homotopy of paths (or path homotopy $f \simeq g\,(\text{via})H$).
(iii) $\simeq$ is an equivalence relation.
(iv) Straight line homotopy in $\mathbb{R}^n$.
(v) The $*$ operation on paths.
(vi) Extension of $*$ operation to homotopy classes of paths and its properties.

2.2. **Fundamental group.**

(i) The space of all homotopy classes of loops based at $x_0 \in X$, $\pi_1(X, x_0)$.
(ii) $(\pi_1(X, x_0), *)$ is a group, the Fundamental group of X based at x_0.
(iii) The isomorphism $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$, where α is a path from x_0 to x_1.
(iv) Simply connected spaces.
(v) In a simply connected space, any two paths having the same initial and end points are path homotopic.

(vi) The homomorphism $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ induced by a continuous map $h : (X, x_0) \to (Y, y_0)$.

(vii) The homomorphism induced by the identity map $(i_X)_*$ is the identity homomorphism on the fundamental group $\pi_1(X, x_0)$.

(viii) If $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ are continuous maps, then $(g \circ f)_* = g_* \circ f_*$.

(ix) If $f : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

2.3. **Covering spaces.**

(i) Evenly covered neighborhoods.

(ii) Covering spaces with examples.

(iii) The standard covering space $p : \mathbb{R} \to S^1$ given by $p(s) = e^{i2\pi s}$.

(iv) If $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ are covering spaces, then $p \times q : \widetilde{X} \times \widetilde{Y} \to X \times Y$ is a covering space.

(v) The covering space $p \times p : \mathbb{R}^2 \to S^1 \times S^1$ of the torus.

(vi) Lifting of continuous maps.

(vii) Let $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ be a covering map. Then any path $f : [0, 1] \to X$ with $f(0) = x_0$ has a unique lift to a path $\widetilde{f}$ in $\widetilde{X}$ with $\widetilde{f}(0) = \widetilde{x_0}$.

(viii) Let $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ be a covering map. Let $H : I \times I \to X$ be a continuous map with $H(0, 0) = x_0$. Then there is a unique lift of H to a continuous map $\widetilde{H} : I \times I \to E$ such that $\widetilde{H}(0, 0) = x_0$. If H is a path homotopy, then so is $\widetilde{H}$.

(ix) Let $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ be a covering space. Let f and g be two paths in X from x_0 to x_1, and let $\widetilde{f}$ and $\widetilde{g}$ be their respective liftings to paths in $\widetilde{X}$ beginning at $\widetilde{x_0}$. If $f \simeq g(via\, H)$, the $\widetilde{f}(1) = \widetilde{g}(1)$ and $\widetilde{f} \simeq \widetilde{g}(via\, \widetilde{H})$.

(x) (Lifting Criterion) Suppose that we have a covering space $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$, and a map $f : (Y, y_0) \to (X, x_0)$ with Y being a path connected and a locally path-connected space.Then a lift $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, x_0)$ exists iff $f_*(\pi_1(Y, y_0)) \leq p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$.

(xi) (Uniqueness of lift) Given a covering space $p : \widetilde{X} \to X$ and a map $f : Y \to X$ with two lifts $\widetilde{f_1}, \widetilde{f_2} : Y \to X$ that agree in one point of Y. If Y is connected, then these two lifts must agree in all of Y.

(xii) If a space X is path-connected, locally path-connected, and semilocally simply-connected, then X has a universal covering space $p : \tilde{X} \to X$.

(xiii) Suppose that a space X is path-connected, locally path-connected, and semilocally simply-connected. Then for every subgroup $H \leq \pi_1(X, x_0)$, there is a covering space $p : X_H \to X$ such that $p_*(\pi_1(X, x_0)) = H$, for a suitably chosen basepoint $\widetilde{x_0} \in X_H$.

(xiv) Isomorphism of covering spaces.

(xv) Suppose that a space X is path-connected, locally path-connected, and semilocally simply-connected. Then two path-connected covering spaces $p_1 : \widetilde{X_1} \to X$ and $p_2 : \widetilde{X_2} \to X$ are isomoprphic via an isomorphism $f : \widetilde{X_1} \to \widetilde{X_2}$ taking basepoint $\widetilde{x_1} \in p_1^{-1}(x_0)$ to a basepoint $\widetilde{x_2} \in p_2^{-1}(x_0)$ iff $p_{1*}(\pi_1(\widetilde{X_1}, \widetilde{x_1})) = p_{2*}(\pi_2(\widetilde{X_2}, \widetilde{x_2}))$.

(xvi) (Classification of covering spaces) Suppose that a space X is path-connected, locally path-connected, and semilocally simply-connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$. Moreover, if the basepoints are ignored, this gives a bijection between isomorphism classes of path-connected covering spaces $p : \widetilde{X} \to X$ and the conjugacy classes of subgroups of $\pi_1(X, x_0)$.

(xvii) Deck transformations and the group $G(\widetilde{X} \xrightarrow{p} X)$ of deck transformations of a covering space $p : \widetilde{X} \to X$.

(xviii) Normal covering spaces.

(xix) Let $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ be path-connected covering spaces of the path-connected, locally path-connected space X, and let $H = p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$. Then:

  (a) This covering space is normal iff $H \trianglelefteq \pi_1(X, x_0)$.
  (b) $G(\widetilde{X} \xrightarrow{p} X) \cong N(H)/H$.

In particular, $G(\widetilde{X}) \cong \pi_1(X, x_0)/H$, if $\widetilde{X}$ is a normal covering. Hence if $p : \widetilde{X} \to X$ is the universal covering, then $G(\widetilde{X} \xrightarrow{p} X) \cong \pi_1(X)$.

(xx) The action of a group G on X, its orbits G_x, and the orbit space X/G.

(xxi) Covering action.

(xxii) If the action of a group G on a space X is a covering action, then:

  (a) The quotient $p : X \to X/G$ is a normal covering space.
  (b) If Y is path-connected, then $G = G(X \xrightarrow{p} X/G)$.
  (c) If Y is path-connected and locally path-connected, then $G \cong \pi_1(X/G)/p_*(\pi_1(X))$.

2.4. **The lifting correspondence and $\pi_1(S^1)$.**

(i) The lifting correspondence.

(ii) Let $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ be a covering map. If $\widetilde{X}$ is a path-connected, then the lifting correspondence $\phi : \pi_1(X, x_0) \to p^{-1}(x_0)$ is surjective. Furthermore, if $\pi_1(\widetilde{X}, x_0) = 0$, then ϕ is bijective.

(iii) Let $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ be a covering map.

(a) Then $p_* : \pi_1(\widetilde{X}, \widetilde{x_0}) \to \pi_1(X, x_0)$ is injective.

(b) The lifting correspondence ϕ induces an injective map

$$\Phi : \pi_1(X, x_0)/H \to p^{-1}(x_0),$$

where $H = p_*(\pi_1(\widetilde{X}, \widetilde{x_0})$. Moreover, Φ is bijective if $\widetilde{X}$ is path-connected.

(c) A loop α based at x_0 lifts to a loop $\widetilde{\alpha}$ based at $\widetilde{x_0}$ iff $[\alpha] \in H$.

(iv) The fundamental group of S^1 is isomorphic to the additive group of integers.

2.5. **Applications of $\pi_1(S^1)$.**

(i) Alternative proof for the fundamental theorem of algebra.

(ii) (Brouwer's fixed point theorem) Every continuous map $h : D^2 \to D^2$ has a fixed point.

2.6. **Seifert-van Kampen Theorem.**

(i) Free groups.

(ii) Universal property of free groups.

(iii) Every group is G the quotient of a free group $F(S)$.

(iv) Every group G has a presentation given by $\langle S|R\rangle$, where S is a generating set, and R is a set of relations.

(v) Free products of groups.

(vi) (Special case of Seifert-van Kampen Theorem) Suppose that $X = U \cup V$, where U and V are open sets of X. Suppose that $U \cap V$ is path-connected and $x_0 \in U \cap V$. Let $i : U \hookrightarrow X$ and $j : V \hookrightarrow X$. Then $\pi_1(X, x_0)$ is generated by $\mathrm{Im}(i_*)$ and $\mathrm{Im}(j_*)$.

(vii) (Seifert-van Kampen Theorem) If X is the union of path-connected open sets A_α each containing the basepoint $x_0 \in X$, and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the natural homomorphism $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ obtained as an extension of the homomorphisms $(j_\alpha)_*$ induced by the inclusion maps $j_\alpha : A_\alpha \hookrightarrow X$, is surjective. If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all the elements of the form $(i_{\alpha\beta})_*(w) * (i_{\beta\alpha})_*(w)^{-1}$, where $(i_{\alpha\beta})_*$ and $(i_{\beta\alpha})_*$ are induced by the inclusion maps $i_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ and $i_{\beta\alpha} : A_\alpha \cap A_\beta \hookrightarrow A_\beta$ respectively. Consequently, Φ induces an isomorphism $\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)/N$.

(viii) Wedge products $\bigvee_\alpha X_\alpha$ of spaces X_α.

(ix) The fundamental group of a wedge product $\bigvee_\alpha S^1$ of circles.

2.7. **Cell complexes.**

(i) Construction of a cell-complex.

(ii) Cell-complex structure on surfaces.

(iii) Products and quotients of cell-complexes.

(iv) The cone CX of a space X.

(v) Subcomplexes and CW-pairs.

(vi) Homotopy Extension Property (HEP) for a pair of spaces (X, A).

(vii) Homotopically equivalent spaces X and Y ($X \simeq Y$).

(viii) If $X \simeq Y$, then $\pi_1(X) \cong \pi_1(Y)$.

(ix) Deformation retraction of a space X onto a subspace A.

(x) If X deformation retracts onto A, then $\pi_1(A) \cong \pi_1(X)$.

(xi) Contractible spaces.

(xii) If (X, A) is a CW-pair, then it has the HEP.

(xiii) If (X, A) is a CW-pair and A is contractible, then the quotient map $X \to X/A$ is a homotopy equivalence.

(xiv) The effect of attaching 2-cells to a path-connected space X.

(xv) Surfaces as cell-complexes.

(xvi) Connection sum operation on surfaces.

(xvii) (The Classification Theorem for closed surfaces) Any closed surface is homeomorphic to the sphere S^2, or the connected sum of finitely many tori, or the connected sum of a finitely many projective planes (i.e. copies of $\mathbb{R}P^2$ or cross-caps).

(xviii) Presentations for the fundamental groups of closed surfaces.

(xix) For every group G, there is a 2-dimensional complex X_G with $\pi_1(X_G) \cong G$.

2.8. **Graphs and free groups.**

(i) Graphs as 1-dimensional cell complexes.

(ii) Trees and maximal trees.

(iii) Every connected graph has a maximal tree.

(iv) Every covering space of a graph is also a graph.

(v) Every subgroup of a free group is free.

(vi) Covering spaces of the figure-eight space.

2.9. **Cayley complexes.**

(i) Cayley graph of a group.
(ii) Cayley complex $\widetilde{X_G}$ of the space X_G.
(iii) $\widetilde{X_G}$ is a simply-connected covering space of X_G, and hence its universal cover.
(iv) When $G = F(a,b)$, $\widetilde{X_G}$ is the tetravalent tree, and when $G = \mathbb{Z}^2$, $\widetilde{X_G} \approx \mathbb{R}^2$.